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APPLICATION OF FUNCTIONAL ANALYSIS IN FLUID-MECHANICS

BY

L. G. N A P O L I T A N O

AERODYNAMICS INSTITUTE, UNIVERSITY OF NAPLES

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APPLICATION OF FUNCTIONAL ANALYSIS IN FLUID-MECHANICS

prepared by

L.G. NAPOLITANO
Aerodynamics Institute, University of Naples

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TABLE OF CONTENT

I. FOREWORD.

The work done during the period covered by the present Final Report constitutes extensions and generalizations of the results found under previous grants and related to the application of functional analysis to Fluid Mechanics.

The specific subject dealt with is that of closed splines which were introduced for the first time by the principal investigator as the appropriate splines to solve interpolation problems commected with the flow fields around airfoils.

This work has lead to the following papers:

I. L.G.Napolitano:A new Characterization of Closed Splines, accepted for publication in the Italian Journal:Aerotecnica Missili e Spazio.

2. L.G.Napolitano:"Closed Smoothing Splines."

The second paper will be subtioned either to the previously mentioned Italian Journal or to an International Journal.

At the time of the writing of the present Final Report the final choice had not been made yet.

The subject Final Report contains, in extenso, the above two papers which constitute Part I and II, respectively, of the report itself.

PART I

A new Characterization of Closed Splines.

A NEW CHARACTERIZATION OF CLOSED SPLINES

I. INTRODUCTION

The author has previously defined and studied new classes of spline functions, referred to as closed splines, which are the proper splines to be used when interpolating data prescribed on a set of points belonging to a closed contour C [1].

The theory of closed splines was based on the Hilbert-space approach [2] and was thus formulated in general terms. Different classes of closed splines can be obtained by appropriate choices of the Hilbert spaces and of the operators acting on them.

Two applications of the general theory have been reported in [1] and [3]. [1] deals with closed splines interpolating values of a function prescribed at a given set of points. [3] deals with Hermite closed splines, i.e. splines interpolating data representing values of a function and of its derivatives prescribed at a given set of points.

Many other classes of closed spline functions can be considered which solve other interpolation problems of practical relevance. Thus, for instance:

i) The prescribed data may represent a linear combination:

$$\alpha_i f(P_i) + \gamma_i f'(P_i)$$

ii) Often, expecially in the case of experimental measurements, one does not know the value of a function at one point but, rather, its average value over a small interval. Thus the prescribed data may be of the form

$$\int_{a_i}^{b_i} f(t)\, W_i(t)\, dt$$

where (ai, bi) are the subintervals to which average values correspond and the weighting factors Wi account for the possible non-uniformity of measurements.

iii) The prescribed data may represent values of a function and of any number of its subsequent derivatives or, also, a linear combination of them .

iv) one may consider trigonometric spline functions (in this case what is being is not previous cases, but the type of operator).

This last is only indicative and is noticeably enlarged if two and three dimensional splines are considered.

For all cases mentioned above the characterization and properties of classical splines (i.e. splines defined over a finite interval)are well established.

Characterization and properties of closed splines (i.e. splines defined over a closed contour), on the contrary, are established only in the two previously mentioned cases of normal [I] and Hermite [3] splines. All other classes of splines remain to be characterized and studied.

The application of the general theory, as formulated in [I] , entails a number of lengthy developments as esemplified by the contents of [I] and [3] . It appearred thus desirable to further examine the general theory with the aim of reformulating it so as to make it possible to utilize, to the laegest possible extent, whatever is already known, and available, from the corresponding classes of classical splines.

This task would also help to shed further light on the nature and types of differences existing between classical and closed splines.

This further development of the general theory of closed splines is presented in this paper for the case in which the space X is an arbitrary space of functions defined over a closed interval.

The new formulation is first applied to the cases already studied in [I], [3] in order to further clarify the essence and actual procedure of the new approach.

Then, as a futher example of application, the new class of closed spline of the type (I) above is characterized and its properties concisely derived. This case shows the extreme usefulness of the new approach and is indicative of its application to the other classes of closed splines mentioned above.

2. RELEVANT RESULTS FROM ABSRACT-SPACE THEORY.

A number of basic results of the Hilbert-space spline theory are needed. They are concisely derived in this section.

Let X,Y and Z be three real Hilbert spaces; $T: X \to Y$; $A: X \to Z$ two linear onto continuous operator and denote by $< , >_W$, the inner product and the orthogonal complement of the space W; and by R(B), N(B), B' the range, the null space and the adjoint of the operator B. R(T) and R(A) are closed in Y and Z, respectively.

Definition and properties of the spline space $S \subset X$ corresponding to T and A hinge on the following equalities:

$$< Ts, Tx >_Y = < \bar{\lambda}, Ax >_Z = < A'\bar{\lambda}, x >_X$$

$$s \in S \quad ; \quad x \in X \quad ; \quad \lambda \in Z \qquad (2.I)$$

where t second one follows from the definition of adjoint operator.

From eqs. (2.I) it follows that:

I) $$< A'\bar{\lambda}, x>_X = 0 \qquad x \in N(T) \tag{2.2}$$

2) $$< Ts, Tx >_y = 0 \qquad x \in N(A) \tag{2.3}$$

Consequently: i) if N(T)+ N(A) is closed in X there will always be at least one $\lambda \in Z$ satisfying eq.(2.I) for any Ax $\in Z$ (existence theorem); ii) if N(T) $\cap$ N(A)= $\{0\}$ this $\bar{\lambda}$ is unique for any given Ax=z$\in$ Z. (Uniqueness theorem) [2].

The equation:

$$Ax=z \tag{2.4}$$

with z an arbitrary but fixed element of Z characterizes the "constraints".

Eq. (2.3) characterizes the spline space S C X associated with the operators (T,A).

When the space Z is finite: Z$\in$ R^n, with its usual inner product, the operator A can be represented as a set of (n) linear and continuous operators Ki on X:

$$Ax= [< K_1, x >_X, -----, < K_n, x >_X]$$

$$< K_i, x >_X = z_i \quad ; \quad z = [z_1, -----, z_n] \in R^n \tag{2.5}$$

and:

$$< A'\bar{\lambda}, x >_X = < \bar{\lambda}, Ax >_Z = \sum_{i=1}^{n} \bar{\lambda}_i < K_i, x >_X$$

$$\bar{\lambda} = [\bar{\lambda}_1, ----, \bar{\lambda}_n] \in Z$$

$$A'\bar{\lambda} = \sum_{i=1}^{n} \bar{\lambda}_i K_i$$

Furthermore:

$$R(A)= Z$$

$$R(A')= [N(A)]^{\perp} = A'(Z)$$

and:

$$\dim R(A) = \dim R(A') = n$$

$$\text{co-dim}\ N(A) = \text{finite} \tag{2.6}$$

$$\dim N(T) = q = \text{finite}$$

The finiteness of $N(T)$ follows from the uniqueness condition $N(T) \cap N(A) = \{0\}$ and the fact that $N(A)$ has finite co-dimension.

From the well known relation between the dimensions of subspaces W_1, W_2

$$\dim(W_1 \cap W_2^{\perp}) = \dim W_1 + \dim(W_1^{\perp} \cap W_2) - \dim W_2$$

and the relations holding for a linear continuous operator with <u>closed</u> ranges:

$$R(B') = [N(B)]^{\perp}; \quad [R(B')]^{\perp} = N(B)$$

it follows that, for $W_1 = [N(A)]^{\perp} = R(A')$; $W_2 = N(T) = [R(T')]^{\perp}$:

$$\dim[R(A') \cap R(T')] = \dim R(A') + \dim[N(A) \cap N(T)] - \dim N(T) = n - q \tag{2.7}$$

From eq.(2.3):

$$\langle T'Ts, x \rangle_y = 0 \qquad x \in N(A)$$

hence $T'Ts \in [N(A)]^{\perp} \cap R(T')$ so that:

$$Ts \in R(T') \cap R(A')$$

from which; upon eq.(2.6)

$$\dim T(S) = n - q \tag{2.8}$$

$$\dim S = n$$

REMARK I

In essence, n is the number of constraints (2.4) which are imposed: i.e. the number of data that are prescribed.

REMARK2

The dimension of T(S) is equal to the number (n) of data prescribed less the dimension (q) of the null space of T .

REMARK 3

The dimension of the spline space S is equal to the number of data that are prescribed.

REMARK 4

All above is quite general. Whether one considers classical or closed splines depends on the set over which the function space X is defined.

3. CHARACTERIZATION OF SPLINES.

The actual procedure most often used to characterize classical splines of a given class can be summarized as follows. Given the (n) operators Ki one finds the function $\psi = Ts \in Y$ by solving equation (2.I)

$$\langle \psi, Tx \rangle_Y = \sum_{i=1}^{n} \bar{\lambda}_i \langle K_i, x \rangle_X \tag{3.I}$$

subject to the conditions of eq. (2.2):

$$\langle A'\bar{\lambda}, x \rangle_X = \sum_{i=1}^{n} \bar{\lambda}_i \langle K_i, x \rangle_X = 0 \quad ; \quad x \in N(T) \tag{3.2}$$

which, in general, are readily expressed as costraints on the $\bar{\lambda}_i$. The number of these constraints is equal to dim N(T), i.e. to (q) [eq.(2.6)] . Thus, as required, by eq.(2.8), ψ contains (n-q) arbitrary paramenters. They constitute, together with the (q) parameters involved in getting from $\psi = Ts$ to s , the n indipendent parameters upon which s depends, as required by eq.$(2.8)_2$.

The solution of eq. (3.I) can often be put in the form [2]:

$$\psi = \sum_{i=1}^{n} \bar{\lambda}_i \psi_i \tag{3.2}$$

where the $\bar{\lambda}_i$ satisfy the (q) constrants (3.2). The questions to be addressed are: Can the known elements ψ_i entering the characterization of classical splines be used to characterize the closed splines? if so, how?

4. CORRESPONDENCE BETWEEN CLASSICAL AND CLOSED SPLINES.

The problem formulated at the end of the preceeding section is posed iff one establishes an appropriate properly correspondence between the spaces and operators correspon-ding to the classical splines and those corresponding to the closed splines.

In this section, all quantities pertaining to closed splines will be denoted with a subscript (c).

It will be assumed that the space X refers to functions defined over the interval $[0,I]$.

Generalization to other cases may, in principle, be possible but will not be cosidered here.

The set of points at which data are prescribed can be characterized by the values τ_i of a non-dimensional parameter τ. These values are such that:

$$0 = \tau_1 < \tau_2 < ---- < \tau_n < \tau_{n+1} = 1$$

For closed splines, the τ_i's are the curvilinear coordinates along an arbitrary contour, C, sufficiently smooth and regular, measured from one of its points ($\tau_1 = 0$) and normalized with respect to its length (τ_{n+1} =I).The common point of coordinates $\tau_1 = 0$, $\tau_{n+1} = 1$ will be referred t as closure point.

A first correspondence is thus established, for both X and X_c are defined, unless an inessential stretching, over the same set $\{\tau\} = [0, 1]$.

The elements x_c of X_c must satisfy continuity requirements at the closure point whose number and nature depends on the type of space X. Hence:

$$X_c \subset X$$

The last requirement for the problem at hand to be well posed is that the classes of splines, be the same. This means that $Y_c \subseteq Y$ $Z_c \subseteq Z$ and the operators Tc, Ac; Tc: $Xc \rightarrow Y$; Ac: $Xc \rightarrow Z$, be the restrictions of the operators T and A to Xc.

The most relevant consequences of these correspondeces will now be analyzed.

To begin with; from $Xc \subset X$ it may follow that $N(Tc) \subset N(T)$ and:

$$\dim N(T_c) = q_c < \dim N(T) = q$$

so that, if $Sc \subset Xc \subset X$ is the space of closed spline functions:

$$\dim T_c(S_c) = n + 1 - q_c > \dim T(S) = n + 1 - q \qquad (4.I)$$

Furthermore, when $q_c < q$ there are also differences in "nature" between ψ and ψ_c because they satisfy different constraints.

To elucidate this point consider the set $\{\psi_i\}$ as a base and denote by $F \subset Y$ the subspace spaimed by this base. On the plansible assumption that the ψ_i 's are indipendent, F is an (n+I)-dimensional space. Consider now the element $f \in F$ defined by:

$$f = \sum_{i=1}^{n+1} \bar{\lambda}_i \psi_i \quad .$$

where the $\bar{\lambda}_i$'s are now arbitrary. Clearly $TS \subset F$ and, more specifically, $f = \psi \in T(S)$ iff the $\bar{\lambda}_i$ satisfy the (q) relations:

$$\sum_{i=1}^{n+1} \bar{\lambda}_i < K_i, x >_X = 0 \qquad x \in N(T)$$

If $N(Tc) \subset N(T)$ and the $\bar{\lambda}_i$ satisfy only the $q_c < q$ relations

$$\sum_{i=1}^{n+1} \bar{\lambda}_i < K_i, x >_X = 0 \qquad x \in N(T_c) \tag{4.2}$$

f belongs neither to $T(S)$ nor to $Tc(Sc)$

The difference between ψ_c and ψ is now evident. It is solely due to the smaller number of constraints imposed on the $\bar{\lambda}_i$ by the smaller dimension of $N(Tc)$ and, does not involve the ψ_i's. Hence it must refledt the different behaviour of ψ and ψ_c at the closure point. For ψ_c the closure point can be any point and thus the features of ψ_c at the closure point are <u>the same</u> as those at any other point. It will be presently seen that this criterion leads to the new characterization of ψ_c which makes uses of all known properties of ψ and ψ_i's .

Embed a subspace od F into an higher dimensional space Fc such that $Tc(Sc) \subset Fc$.

Only one of the two elements ψ_1 or ψ_{n+1} of the basis of F is relevant, as it can be inferred from the continuity conditions at the closure point.

Thus the base of Fc contains the first n element ψ_i. On the other hand, in order for $Tc(Sc) \subset Fc$, the dimension p of Fc must be such that once the qc constraints (4.2) and the remaining (q-qc) satisfied by ψ but not ψ_c are endorsed, one obtains a (n+I-qc) dimensional space, as required by eq. (4.I).

Hence p-qc-(q-qc)=n+I-qc, i.e. p=n+(q-qc-I).

The element $f_c \in F_c$ can thus be formally written as:

$$f_c = \sum_{i=1}^{n} \bar{\lambda}_{ic} \psi_i + \sum_{j=0}^{q-q_c} \beta_j \tilde{\psi}_j \tag{4.3}$$

where the $\tilde{\psi}_j$ must be such that $T_c S_c \subset F_c$ and can usually be found, by inspection, once the expression of the ψ_i's is known.

The new formulation for the characterization of closed spline is obtained by combining eq.(4.3) with the previously mentioned criterion.

Let S any classical spline space defined over the interval $[0,1]$ and call interior point Pi any point within this interval and extremal the left and right extreme of the interval.

The characterization of an element $s \in S$ defines the function ψ_i and the properties of Ts at interior points. In particular one knows:

1) the number and types of discontinuities at interior points.

2) the number and types of continuity properties at the interior points.

Denote by $\{D\}$ the set of interior point discontinuities and by $\{C\}$ the set of interior point continuity properties. Then the following characterization of closed splines of the same class holds:

<u>New characterization of closed splines.</u>

For any class of classical splines defined over a closed interval the corresponding closed spline can be expressed as:

$$\psi_c = T_c s_c = \sum_{i=1}^{n} \lambda_i \psi_i + \sum_{j=0}^{q-q_c} \beta_j \tilde{\psi}_j \tag{4.4}$$

where ψ_i is given by the classical spline, $\tilde{\psi}_j$ is inferred by inspection and the $(n+q-q_i + 1)$ coefficients λ_i and β_j are determined by imposing: at the closure point:

1) the congruence of the discontinuites of the set $\{D\}$.

2) the vanishing of the discontinuities corresponding to the set $\{C\}$.

Condition (1) can be expressed for an arbitrary function h discontinuous at the points Pi:
as (see [3] for the proof):

$$\sum_{i=1}^{n} \left[\Delta_i h + \delta h(P_i)\right] = 0 \tag{4.5}$$

where:

$$\Delta_i h = h(\tau_{i+1}^{+}) - h(\tau_i^{+})$$

$$\delta h(P_i) = h(\tau_{i+1}^{+}) - h(\tau_i^{-})$$

REMARK 1

Once $\psi_i = \Sigma c$ sc is known, obtaining sc is a trivial matter.

REMARK 2

For classical splines, conditions (2) are, by definition, not satisfied.

5. EXAMPLES OF APPLICATION OF THE NEW FORMULATION.

The classes of splines for which the closed ones have already been derived [I], [3] will be dealt with first to help clarifying the procedure to be used.

5.I Normal splines [I].

Spaces and operators are chosen as:

$$X = H^q \; ; \quad Y = H^0 \; ; \quad T = \frac{d^q}{(d\tau)^q} = D_q$$

$$< K_i, x >_{H^q} = x(\tau_i) = r_i \quad ; \; i \in [1, n]$$

where H^q is the Hilbert space of real functions having square-intergrable q^{th} derivatives.

From [2] one gets

i)

$$\psi_i = \frac{(\tau - \tau_i)_+^{q-1}}{(q-1)!} \quad ; \; 1 \le i \le n$$

with:

$$(\tau - \tau_i)_+ = \begin{cases} (\tau - \tau_i) & \text{if } (\tau - \tau_i) \ge 0 \\ 0 & \text{if } (\tau - \tau_i) < 0 \end{cases}$$

ii)

$$\tilde{\psi}_j = \tau^j / j!$$

iii) the derivative of order (q-I) of ψ is discontinuous (set $\{D\}$ of dimension I)

iv) ψ and its derivatives up to order (q-2) are continuous (set $\{C\}$ of dimension q-I)

For closed splines defined over the closed contour C dim N(Tc)=dim N H^q (C)=I [ ], hence q_c =I.
Thus, upon eq.(4.4)

$$s_c^{(q)}(\tau) = \sum_{j=0}^{q-1} \beta_{q+j} \frac{\tau^j}{j!}$$

with the (r+q) coefficients satisfying the conditions:

$$\{D\} \rightarrow \quad \sum_{i=1}^{n} \lambda_i = 0 \tag{5.I}$$

$$\{C\} \rightarrow \quad \delta\psi_c^{(k)} = \psi_c^{(k)}(0^+) - \psi_c^{(k)}(1^-) = 0 \quad ; 0 \le k \le q-2$$

Remarks

For conventional splines:

. The conditions analogous to the first of eqs.(5.I)

$$\sum_{i=1}^{n+1} \bar{\lambda}_i = 0 \tag{5.2}$$

expresses the discontinuity $\bar{\lambda}_{n+I}$ at the point Pn+I in terms of those ($\bar{\lambda}_i$) at all other points.

. The set of equations due to the constraint $x \in N(T)$ leads to the conditions:

$$\psi^{(k)}(0^+) = \psi^{(k)}(1^-) = 0. \qquad 0 \le k \le q-1 \tag{5.3}$$

. Equation (5.2) is also a congruence condition since outside the interval $[0,I]$ ψ is zero. Indeed:

$$\bar{\lambda}_1 = \delta\psi^{(q-1)}(P_1) = \psi^{(q-1)}(0^+)$$

$$\bar{\lambda}_i = \delta\psi^{(q-1)}(P_i) = \psi^{(q-1)}(\tau_i^+) - \psi^{(q-1)}(\tau_i^-) \quad ; 2 \le i \le n$$

$$\bar{\lambda}_{n+1} = \delta\psi^{(q-1)}(P_{n+1}) = -\psi^{(q-1)}(\tau_{n+1}^-)$$

from which eq.(5.I) follows since $\psi^{(q-1)}$ is constant in each sub-interval: $\psi^{(q-1)}(\tau_{i+1}^-) = \psi^{(q-1)}(\tau_i^+)$.

5.2 Hermite splines [3]

Space and operators are now chosen as:

$$X = H^q \quad ; \quad Y = H^0 \quad ; \quad T' = D_q$$

$$\langle K_i, x \rangle_X = x(\tau_i) = z_i \quad ; \quad \langle K_i^{(1)}, x \rangle_X = Dx(\tau_i) = z_i' \; ; \; i \in [1,n]$$

so that now dim$\mathbf{Z} = 2n$ (see remark I,Sect.2).

From [2] one gets:

i) $\psi_i = (\tau - \tau_i)_+^{q-1}/(q-1)!$; $\psi_i^1 = (\tau - \tau_i)_+^{q-2}/(q-2)!$; $i \in [1,n]$

ii) $\tilde{\psi}_j = \tau_j / j!$

iii) The derivatives of order (q-I) and (q-2) are discontinuous (set $\{D\}$ of dimension 2).

iv) ψ and its derivatives of order up to (q-3) are continuous (set $\{C\}$ of dimension (q-2) .

Thus, upon eq.(4.4):

$$\psi_i = s_i^{(q)}(\tau) = \sum_{j=0}^{q-1} \beta_{q+j} \frac{\tau^j}{j!} + \sum_{i=1}^{n} \left[\frac{\lambda_i (\tau - \tau_i)_+^{q-1}}{(q-1)!} - \lambda_i^1 \frac{(\tau - \tau_i)_+^{q-2}}{(q-2)!} \right]$$

The (2n+q) coefficients satisfy the q equations:

$$\{D\} \to \sum_{i=1}^{n} \lambda_i = 0 \quad ; \quad \beta_{2q-1} + \sum_{i=1}^{n} [-\lambda_i \tau_i + \lambda_i^1] = 0$$

$$\{C\} \qquad \delta \psi_i^{(k)} = 0 \qquad\qquad 0 \le k \le q-3$$

The second equation follows from eq.(4.5) [3] .

For conventional splines the λ_i satisfy the (q) conditions [2] :

$$\sum_{i=1}^{n+1} [\lambda_i \tau_i^j - j \lambda_i^1 \tau_i^{j-1}] = 0 \qquad 0 \le j \le q-1$$

which lead to the same conditions (5.3) at the extremal points.

The space of Hermite closed splines is of dimensions(2n).

One more class of closed splines will now be considered.It is new insofar as it has not been introduced before. Its classical counterparts is well known. The new formulation will be used. Results are stated concisely.

5.3 <u>Data are linear combinations of values of a function and its first derivative</u>.

The spaces and operators are defined as:

$$H = H^q(C) \; ; \quad Y = H^0(C) \; ; \quad T = D_q$$

$$\langle K_i, x \rangle_{H^q} = \alpha_i x(\tau_i) + \gamma_i x'(\tau_i) \quad ; \quad i \in [1, n]$$

where the constants α_i, γ_i are not all vanishing. From [3] :

i) $$\psi_i = \frac{\alpha_i (\tau - \tau_i)_+^{q-1}}{(q-1)!} - \beta_i \frac{(\tau-\tau_i)_+^{q-2}}{(q-2)!}$$

ii) As the operator T is the same as in the previous two cases the $\tilde{\psi}_j$ will be the same.

iii) The derivatives of order (q-I) and (q-2) are discontinuous (set $\{D\}$ of dimension 2) and are related by the:

$$\alpha_i \, \delta s^{(2q-2)}(P_i) + \gamma_i \, \delta s^{(2q-1)}(P_i) = 0$$

iv) ψ and its derivatives up to order (q-3) are continuous (set $\{C\}$ of dimension (q-2)).

Thus, upon eq.(4.4)

$$\psi_i = s_i^{(q)}(\tau) = \sum_{j=0}^{q-1} \beta_{q+j} \frac{\tau^j}{j!} + \sum_{i=1}^{n} \lambda_i \psi_i$$

where the (n+q) the coefficients β_j, λ_i satisfy the equations:

$$\{D\} : \quad \sum_{i=1}^{n} \lambda_i = 0 \quad ; \quad \sum_{i=1}^{n} \lambda_i [\alpha_i \tau_i + \gamma_i] = 0$$

$$\{C\} \qquad \delta \psi_c^{(k)} = 0 \qquad\qquad 0 \le k \le q-3$$

The relevant properties of these classes of closed splines are:

I) σ_i is a polynomial of degree (2q-I) in each open interval $(0=\tau_1, \tau_2), \ldots, (\tau_i, \tau_{i+1}), \ldots, (\tau_n, 1)$.

2) σ_c is continuous on C together with its first (2q-3) derivatives

3) at each point (τ_i) the discontinuities of the (2q-2)-th and (2q-I) -th derivatives are such that:

$$\alpha_i \, \delta_i \, \sigma_c^{(2q-2)} + \gamma_i \, \delta_i \, \sigma_c^{(2q-1)} = 0 \qquad \forall i \in [1, n]$$

4) σ_c is such that:

$$\alpha_i \, \sigma_c(\tau_i) + \gamma_i \, \frac{d\sigma_c}{d\tau}(\tau_i) = 0 \quad ; \forall i \in [1, n]$$

The set of functions of H^q(C) satisfying the first three conditions contitutes the n-dimensional closed spline space Sc.

The uniqueness conditions read [3] :

$$\alpha_i \, x(\tau_i) + \gamma_i \, \frac{dx_i}{d\tau} = 0 \qquad \forall i \in [1, n]$$
$$x^q(\tau) = 0 \qquad . \qquad \tau \in C$$

imply $x(\tau) \equiv 0$.

6. Concluding remarks.

A new formulation for the characterization of closed splines has been derived.

This new formulation makes it possible to use a number of facts already known from the theory of classical splines and helps gaining a deeper understanding of the differences between the two types of splines.

The applicatio of this new formulation is esemplified for three classes of splines. Two of them had already been derived [I], [2] with the direct approach. They have been considered in order to further clarify the essence of the new formulation.
The ~~last~~ third class has been deduced here for the first time.

The new formulation by passes long and tedious developments and makes it almost immediate to deduce the closed spline counterpart of known classical splines.

The new formulation concerns only the characterization of closed splines. All other relevant properties (existence, uniqueness; extremal properties equivalente with a minimum problem) are those expounded and discussed in the original derivation of the theory [I] and have consequently not been repeated here.

7. References

I-L.G.Napolitano, V. Losito:"The closed spline functions" Comp. Methods in Applied Mechanics and Energineering Vol I3 (I978) pp 335-350.

2- J.P. Laurent: Approximation et Optimization, Hermam, Paris, I972.

3-L.G.Napolitano: "Hermite Closed spline Functions" Aerot. Missili +Spazio, Vol 7,NoI (I978)pp30-36.

PART II

Closed Smoothing Splines.

CLOSED SMOOTHING SPLINES

I. INTRODUTION

In computationsl aerodynamicsit is often necessary to solve interpolting problems related to airfoils, i.c. to closed curves.

Using classical spline functions to interpolate airfoil's ordinates prescribed on a finite set of points is unsatisfactory on many important acconuts, as fully discussed in [I]. Similar situations arises in all other interpolating problems. The case of Hermite spline functions (the data to be interpolated represent values of the functions and of its first derivate at a given set of points) was discussed in [2].

The main short comings can be briefly described, in general terms, as follows. When the data to be interpolated are prescribed on a set of $(n+1)$ points Q_i belonging to a closed curve (e.g. the airfoil) and conventional splines are used, then the first and last points must be taken as coincident $Q_{n+1}=Q_1$ (closure point-usually the airfoil's trailing edge). As a consequence:

i) unwanted (and unnecessary) discontinuities are introduced at the closure point;

ii) a number of derivaties of the unterpolating spline vanish at the closure point and this may turn out to be too penalizing in practical applications, especially with splines of low order;

iii) the degree of smoothness of the interpolating function is neither quantificable nor uniform over the closed contur with the accuracy being very poor at the closure point;

iiii) the interpolation function does not satisfy any minimization problem and thus all the advantages comected with this fact are lost.

To overcome all these shortcomings the author developed a theory for a new classes of splines, referred to as <u>closed splines</u> [I], based on the general abstract-space spline theory detailed in [3].

The abstract space approach presents the following main advantages: i) existence and uniqueness can be proved; ii)general characteri= zations of the splines can be given; iii) the equivalence between an interpolation problem and a minimization problem can be establisned; iiii) extremal and other important properties of the splines can be demostrated.

This paper addresses, with the same approch, another important class of problems of frequent occurrence in computational aerodynamics: namely the one solved by smoothing splines.

To clarify the issues involved, consider the case in which the data to be interpolated represent the values z_i of a function at a given set of points Q_i.

In many cases the data prescribed are not exact but approximate, i.e. affected by either errors or uncertainties. It thus makes little sense to have the interpolating function f assume exactly the values z_i at the points Q_i and one would rather like to achieve a suitable compromise between the "smoothmess" of the interpolating function and the "approximation" of the prescribed data z_i. The classes of splines achieving this compromise are called "smoothing" splines.

The formulation of smoothing problems solved by classical smoothing splines and their properties are well known [3] and can be summarized as follows.

Given a closed interval, reduced through suitable normalization to $[0,1]$; $(n+1)$ points $Q_i(\tau_i)$ with:

$$0 = \tau_1 < \tau_2 < ---- < \tau_n < \tau_{n+1} = 1$$

and $(n+1)$ real numbers z_i $(1 \le i \le n+1)$, the smoothing spline function of order $q \le (n+1)$ cirresp onding to the set $\{\tau_i, z_i, \rho\}$, where ρ is a positive constant, is the unique function $\tilde{\sigma} \in H^q[0,1]$ which solves the following minimum problem:

$$\min_{\rho \in H^q} \left\{ \int_0^1 [f^{(q)}(\tau)]^2 d\tau + \rho \sum_{i=1}^{n} [f(\tau_i) - z_i]^2 \right\} \qquad (1$$

where $H^q[0,1]$ is the Hilbert space of real functions $f(\tau)$ defined on $[0,1]$ and having a square-intergrable q^{th} derivative $f^{(q)}$. The space H^q quantifies the degree of smoothness, the coefficient ρ characterizes the relative importance that one assignes to the smoothness and the approximation of data. (More generally, different weights ρ_i can be prescribed for each point).

When one sets $f(\tau_i) = t_i$ the minimum problem (I.I) reduces to the one pertaining to the interpolating splines [I,3] and, for $q=(n+1)$ the interpolating spline is the unique polinominal of degree $(n+1)$ passing through the points $f(\tau_i)=t_i$.

Smoothing and interpolating splines belong to the same subspace $S \subset H^q$ of real functions $s(\tau)$ defined on $[0,1]$ and such that [3]:

a) $s(\tau)$ is a polynomial of degree (2q-I) in each of the open intervals (τ_i, τ_{i+1}) $(1 \leq i \leq n+1)$

;

b) $s(\tau)$ and its first (2q-2) derivatives are continuous on $[0,1]$

c) the derivatives of $s(\tau)$ from orders q to (2q-I), included, vanish at the points $Q_1(\tau_1=0)$ and Q_{n+1} $(\tau_1=1)$.

The interpolating spline for the set $\{t_i\}$ is the unique element $\hat{\sigma}$ of S such that: $\hat{\sigma}(\tau_i)= t_i \quad , \forall i \in [1, n+1]$.

The smoothing spline $\tilde{\sigma}$ for the set $\{t_i, \rho\}$ is the unique element of S such that:

$$\rho\,[t_i - \tilde{\sigma}(\tau_i)] = (-1)^q\, \delta_i\, \tilde{\sigma}^{(2q-1)} \quad ; \forall i \in [1, n+1]$$

where:

$$\delta_i\, \tilde{\sigma}^{(2q-1)} = \tilde{\sigma}^{(2q-1)}(\tau_i^+) - \tilde{\sigma}^{(2q-1)}(\tau_i^-)$$

is the discontinuity (jump) of the (2q-I)-th derivative of $\tilde{\sigma}$ at the poin Q_i.

Since the smoothing spline belongs to the same space S as the interpolating spline, it is evident that when the set of points belong to a closed contour, smoothing problems exhibit all the previously mentioned shortcomings.

Thus the need for developing a theory of closed smoothing splines exists and is substantiated by the same consideration that were made in [1,2] when dealing with interpolation problems on closed curves.

In this paper the general theory of closed smoothing splines is developed. It forms the basis for the subsequent consruction and study of different classes of smoothing splines(normal, Hermite, Fourier,Trigonometric, local averange and so forth [3]).

As a first example of application of the general theory, the normal closed smoothing splines, corrisponding to the problem discussed above, are derived and their propertіes studied.

As in [1] and [2], the presentation does not follow the logical order. To facilitate a more wide spread comprehension and use of normal closed, smoothing splines their definition, characterization and properties are first stated without proof in section (2). The general Hilbert space theory of closed smoothing splines is then developed in sections 3.

The last section 4 presents the proof of the statements made in Section 2 concerning existence, uniqueness, characterization, extremal and other relevant properties.

2. NORMAL CLOSED SMOOTHING SPLINES.

2.I DEFINITION

Let C be a sufficiently smooth and regular closed contour. Denote by τ the curvilinear coordinate along C measured from an arbitrary initial point Q_1 and normalized with respect to the lengthof C so that $0 \leq \tau \leq 1$. The point Q_1 will be referred to as the closure point of the contour and it is characterized by either values $\tau = 0^+$ and $\tau = 1^-$ of τ.

Consider (n) arbitrary successive points Q_i ($1 \leq i \leq n$) on C, let τ_i be their curvilinear coordinates with:

$$0 = \tau_1 < \tau_2 < \tau_3 ---- < \tau_n < 1$$

and prescribe n real numbers (t_i) and a positive constant ρ.

The closed smoothing spline $\sigma(\tau)$ of degree (q) corresponding to the (n) conples (τ_i, t_i) ($1 \leq i \leq n$) is defined as the unique element of H^q(C) such that:

$$\oint_C [\sigma^{(q)}(\tau)]^2 d\tau + \rho \sum_{i=1}^{n} [\sigma(\tau_i) - t_i]^2 =$$

$$= \min_{f \in H^q(C)} \left\{ \oint_C [f^{(q)}(\tau)]^2 d\tau + \rho \sum_{i=1}^{n} [f(\tau_i) - t_i]^2 \right\} \qquad (2.I),$$

In order for $\sigma \in H^q$(C) to be such a closed smoothing spline it is necessary and sufficient that:

a) $\sigma(\tau)$ be a polinomial of degree (2q-I) in each open interval:

$$(0 = \tau_1, \tau_2) , --- , (\tau_i , \tau_{i+1}) , ---- , (\tau_n, 1)$$

b) $\sigma(\tau)$ be continuous on C together with its first (2q-2) derivatives, i.e

$$\left.\begin{array}{l} \sigma^{(k)}(0^+) = \sigma^{(k)}(1^-) \\ \sigma^{(k)}(\tau_i^+) = \sigma^{(k)}(\tau_i^-) \ ; \ \forall i \in [2, n] \end{array}\right\} \forall k \in [0, 2q-2] \qquad (2.2)$$

c) $\sigma(\tau)$ be such that:

$$\rho[\tau_i - \sigma(\tau_i)] = (-1)^q \lambda_i = (-1)^q \delta_i \sigma^{(2q-1)} \quad ; \forall i \in [1, n] \tag{2.3}$$

where: $\delta_i \sigma^{(2q-1)} = \sigma^{(2q-1)}(\tau_i^+) - \sigma^{(2q-1)}(\tau_i^-)$

denotes the discontinuity of the (2q-I)-th derivative at the point Q_i, the discoutinuity at the closure point being defined as: $\delta_1 \sigma^{(2q-1)} = \sigma^{(2q-1)}(0^+) - \sigma^{(2q-1)}(1^-)$

The set of functions $s \in H^q(C)$ satisfying the first two conditions constitute the space S_c of the closed spline functions correspon= ding to the set $\{\tau_i\}$. S_c is a subspace of $H^q(C)$ of dimension (n). Interpolating and smoothing spline functions relative to the set $\{\tau_i\}$ are subspace of the same space $S_{c(C)}$

For Interpolating splines ~~are~~ replaced by the conditions $\sigma(\tau_i) = \tau_i \ (1 \leq i \leq n)$.

2.2 CHARACTERIZATION

Introduce the function [3]:

$$(\tau - \tau_i)_+^{2q-1} = \begin{cases} (\tau - \tau_i)^{2q-1} & \text{if } (\tau - \tau_i) \geq 0 \\ 0 & \text{if } \tau - \tau_i < 0 \end{cases} \tag{2.4}$$

An element $s \in H^q(C)$ belongs to the subspace S_c of closed spline functions corresponding to $\{\tau_i\}$ iff it is representable as:

$$s(\tau) = \sum_{j=0}^{2q-1} \beta_j \frac{\tau^j}{j!} + \sum_{i=1}^{n} \lambda_i \frac{(\tau - \tau_i)_+^{2q-1}}{(2q-1)!} \tag{2.5}$$

where the n coefficient λ_i and the2q coefficients β_j satisfy the following (2q) equations [I]

$$\sum_{i=0}^{n} \lambda_i = 0$$

$$-\delta_1 s^{(k)} = \sum_{j=0}^{2q-2-k} \left[\frac{\beta_{j+k+1}}{(j+1)!} - (-1)^{j+k} \sum_{i=1}^{n} \lambda_i \frac{\tau_i^{2q-k-j-1}}{j!(2q-k-1-j)!} \right] = 0 \tag{2.6}$$

$$0 \leq k \leq 2q-2$$

The function defined by eqs. (2.4) is continuous on C together with its first (2q-2) derivatives. Its (2q-I)-th derivative is discontinuous at $\tau=\tau_i$, the discontinuity being equal to (2q-I)! [3].

The (2q-I)-th derivative of $s(\tau)$ is given by:

$$s^{(2q-1)}(\tau)=\beta_{2q-1}+\sum_{i=1}^{n}\lambda_i(\tau-\tau_i)_+^0$$

Hence it is piecewise constant in each open sub-interval and:

$$s^{(2q-1)}(\tau)=\beta_{2q-1}+\sum_{j=1}^{i}\lambda_j \quad \text{in} \quad (\tau_i,\tau_{i+1})\,;\ 1\le i\le n-1$$

$$s^{(2q-1)}=\beta_{2q-1} \quad \text{in} \quad (\tau_n,1)$$

Thus the (n) coefficients λ_i represent the values of the discontinuity of $s^{(2q-1)}$ at the points Q_i:

$$\lambda_i=\delta_i s^{(2q-1)}=s^{(2q-1)}(\tau_i^+)-s^{(2q-1)}(\tau_i^-)$$

Since for $i=1$ it is $\tau_1^+=0$ and $\tau_1^-=1^-$:

$$\delta_1 s^{(2q-1)}=s^{(2q-1)}(0^+)-s^{(2q-1)}(1^-)=\lambda_1$$

The closure point Q_1 is thus in no-way differentiated from the other (interior) points.

Given (n) arbitrary real numbers (t_i) and a real constant ρ there is a unique closed smoothing spline corresponding to the sets $\{t_i\}$, $\{\tau_i\}$, ρ such that:

$$\rho\,[t_i-s(\tau_i)]=(-1)^q\lambda_i=(-1)^q\delta_i s^{(2q-1)}\,;\quad 1\le i\le n \tag{2.8}$$

Consequently the system of $(n+2q)$ equations(2.6),(2.8), linear in β_j and λ_i; admits of a unique solution. When the $(n+2q)$ coefficients appearing the eq (2.5) are so obtained, $s(\tau)$ becomes the unique smoothing spline $\sigma(\tau)$ solving problem (2.I).

When eqs. (2.8) are replaced by the equations:

$$s(\tau_i)=t_i \qquad 1\le i\le n$$

the solution of the system formed by these equations and eqs.(2.6) is still unique.The new coefficients , thus derived, when sostitued in eq. (2.5), define the unique interpolating closed spline $\hat{\sigma}(\tau)$ which solves the minimum problem [I]:

$$\min_{f\in H^q}\oint[f^{(q)}(\tau)]^2d\tau\,;\qquad f(\tau_i)=t_i\,;\quad 1\le i\le n$$

formally obtained from the problem (2.I) by setting $\hat{\sigma}(\tau_i)=t_i$; $f(\tau_i)=t_i\,;\ \forall i\in[1,n]$

2.3. PROPENTIES

The closed smoothing spline σ corresponding to the set $\{\tau_i, t_i, \rho\}$ has the properties a),b),c), described in section(2.I).

Plus,, obviously, all properties of closed splines. Thus, for instance, the first of equations (2.6) represents the congruence conditions for the discontinuities $\delta_i s^{(2q-1)}$ [2], for any element $s \in S_c$ and any function $f \in H^q(C)$ the following relation holds [I],

$$\oint_C s^{(q)}(\tau) f^{(q)}(\tau)\, d\tau = (-1)^q \sum_{i=1}^{n} \lambda_i f(\tau_i) \tag{2.9}$$

In particular:

$$\oint_C s^{(q)} f^{(q)}\, d\tau = 0 \qquad \forall f \in I_0 = \{ f \in H^q(C) \mid f(\tau_i) = 0,\ \forall i \in [1,n] \} \tag{2.10}$$

When s is a closed smoothing spline σ, eq. (2.3) holds and eq. (2.9) becomes:

$$\oint_C \sigma^{(q)} f^{(q)}\, d\tau = \rho \sum_{i=1}^{n} [t_i - \sigma(\tau_i)] f(\tau_i)$$

and eq. (2.I0) still holds for $s = \sigma$.

The minimum problem solved by $\sigma(\tau)$ is a particular case of the following more general extremal properties.

Given (n) arbitrary but fixed (τ_i), if σ is the unique element of S_c such that:

$$\rho\, [t_i - \sigma(\tau_i)] = (-1)^q \delta_i \sigma^{(2q-1)}$$

then:

a) for any $f \in (C)$:

$$\oint_C [\sigma^{(q)} - f^{(q)}]^2 d\tau + \rho \sum_{i=1}^{n} [f(\tau_i) - \sigma(\tau_i)]^2 = \min_{s \in S} \left\{ \oint_C [s^{(q)} - f^{(q)}]^2 d\tau + \rho \sum_{i=1}^{n} \left[f(\tau_i) - t_i + \frac{(-1)^q}{\rho} \delta_i s^{(2q-1)} \right]^2 \right\} \tag{2.11}$$

and σ is the unique element having this propety.

The original minimum problem(2.I) follows from eq.(2.I2) for s=o.

2.4. COMPARISON WITH CLASSICAL SMOOTHING SPLINES.

The classical smoothing spline over ($n+1$) points Q_i is representable as [3]:

$$\tilde{\sigma}(\tau) = \sum_{j=0}^{q-1} \tilde{\beta}_j \frac{\tau^j}{j!} + \sum_{i=1}^{n+1} \frac{\tilde{\lambda}_i (\tau-\tau_i)_+^{2q-1}}{(2q-1)!} \tag{2.I3}$$

The ($n+q$) coefficients $\tilde{\beta}_j$, $\tilde{\lambda}_j$ represent the unique solution of the following system of equations:

$$\begin{aligned} &\sum_{i=0}^{n+1} \tilde{\lambda}_i \tau_i^k = 0 \quad &; 0 \le k \le (q-1) \\ &\rho [t_i - \tilde{\sigma}(\tau_i)] = (-1)^q \tilde{\lambda}_i \quad &; 1 \le i \le n+1 \end{aligned} \tag{2.I4}$$

The space $\tilde{S}$ of classical splines is the set of all $\tilde{s}$ of the form given by eq. (2.I3) with the coefficients satisfying only the first (q) eqs.(2.I4).

When eqs. (2.I4) are used for the set of points belonging to a closed contour, $Q_1 = Q_{n+1}$ and $t_1 = t_{n+1}$. Since the coefficients $\tilde{\beta}_j$, $\tilde{\lambda}_j$ in eq. (2.I3) are uniquely determined it follows that:

i) at the closure point ($Q_1 = Q_{n+1}$) the condition $\tau_1 = t_{n+1}$ guaraktees only the continuity of $\tilde{\sigma}(\tau)$: its derivatives, up to the order $(q-1)$ included, will not be continuous(unless the problem's data satisfy very paricular synmetry conditions) and their discontinuities are uniquely determined.

ii) the subsequent derivatives, up to the order (2q-2), are continuous but vanish identically (see sect. I.)

This has to be contrasted with the closed smoothing spline which is continuous throught together with its first (2q-2) derivatives, the properties at the closure point being in no-way different from those at the other points.

The classical smoothing spline has ($q+n+1$) parameters: if one introduces ($q-1$) additional parameters to impose the missing continuity of the ($q-1$) derivatives at the closure point, one would get a function with a total of($q+n+1$) + ($q-1$)=(2q + n) parameters. These are indeed the number of parameters appearing in the definition, eq.(2;5), of the closed smoothing spline. This rather naive approach does not however indicates how the additional parametersshould be introduced, what systemof equations should be formulated for their unique determination(clearly the greatest part of eqs. (2.I4) cannot be retained) and, most important of all, no hints could be obtained as to how to determine the properties of these n functions.

The obstract approach, leading to the results presented in sections 2.I through 2.3, provides in the more natural and correct maimer the answers to all questions above.

3. THE SMOOTHING SPLINE THEORY IN ABSTRACT SPACE

3.1 ARBITRARY HELBERT SPACES.

The needed basic results of the Hilbert space formulation of smoothing spline function theory are summarized here for ready and convenient reference (see [3] for greater details).

Let X,Y,Z, be three real Hilbert spaces with norms $\|\cdot\|_X$; $\|\cdot\|_Y$; $\|\cdot\|_Z$; and let $T: X \to Y$; $A: X \to Z$ be two linear continuous operators which, without any loss of generality, are supposed to be onto.

Given any fixed $z \in Z$ and any real constant $\rho > 0$ consider the following minimum problem

$$\|T\sigma\|_Y + \rho\|A\sigma - z\|_Z = \min_{x \in X}\{\|Tx\|_Y + \rho\|Ax - z\|_Y\} \qquad (3.1)$$

The element $\sigma \in X$, if it exists, is called the smoothing spline corresponding to (T, A, z, ρ).

Introduce the Helbert space P, cartesian product of Y and Z endowed with the scalar product:

$$\langle p_1, p_2\rangle_P = \langle y_1, y_2\rangle_Y + \rho\langle z_1, z_2\rangle_Z \qquad (3.2)$$

where $\langle , \rangle_F$ denotes the inner product in the Helbert space F.

Then if $L: X \to P$ is the linear continuos operator defined by:

$$Lx = [Tx, Ax] \in P \qquad (3.3)$$

and if:

$$p_0 = [0, z] \in P$$

where 0 denotes the null element of Y, the minimum problem (3.1) can also be formulated in the space P as:

$$\|L\sigma - p_0\|_P = \min_{x \in X}\|Lx - p_0\|_P \qquad (3.4)$$

where $\|\cdot\|_P$ denotes the norm of P associated with scalar product (3.2).

The followinq existence and uniqueness theorem holds.

THEOREM I. The solution of problem (3.1) exists for any $z \in Z$ iff $N(T)+N(A)$ in closed in X and is nuique iff, in addition, $N(T) \cap N(A) = \{0\}$. Here N(B) denotes the null space of the operator B.

The characterization of the smoothing spline function is given in the following theorem.

THEOREM 2. Under the hypothesis of theorem I, $\sigma \in X$ is the smothinq spline corresponding to(T,A,Z, ρ) iff:

$$< L\sigma - p_0, Lx >_P = 0 \qquad \forall x \in X \tag{3.5}$$

Thus the smoothing spline space S is the subspace of X defined by:

$$S = \{ s \in X \,/\, < Ls - p_0, Lx >_P = 0, \ \forall x \in X \} \tag{3.6}$$

or, equivalently, on account of the definition of the adjoint of L; by:

$$S = \{ s \in X \,/\, L'Ls \in R(A') \} \tag{3.7}$$

where the prime denotes, here and in what follows, the adjoint of an operator; $R(B)$ is the range of the operator B and the adjoint L' of L is defined by:

$$L'p = T'y + \rho A'z \ ; \qquad p = [y, z] \in P \tag{3.8}$$

The second definition follows from the first one on account of the oroperties of adjoint operators and of the fact that $L'p_0 = \rho A'z \in R(A')$:

$$< Ls - p_0, Lx >_P = < L'Ls - L'p_0, x >_P = 0 \qquad \forall x \in P$$

$$\Rightarrow \quad L'Ls = L'p_0$$

From the definition (3.7) it also follows that:

$$L'L(S) = R(A') \tag{3.9}$$

REMARK I.

It can be realy shown [3] that S does not depend on (ϱ) and it thus coincides with the space S^* of the spline fonctions defined by the operators T and A.

THEOREM 3. For any fixed $z \in Z$ there exists a unique element $\sigma \in X$ such that:

$$A\sigma + \frac{1}{\varrho} B\sigma = z$$
$$B : S \to Z \quad ; \quad B = (A')^{-1} T'T \tag{3.10}$$

This space S^* can be characterized as [3] :

$$S^* = \{ s \in X \mid \langle Ts, Tx \rangle_Y = 0 \; ; \; \forall x \in N(A) \}$$

where N (•) denotes the null space. Hence $s \in S^*$ iff:

$$T'Ts \in [N(A)]^{\perp} = \mathcal{R}(A')$$

Hence:

$$T'T(S^*) = R(A')$$

and, upon eq. (3.3), $S^* \equiv S$ since $L'Ls = T'Ts + \varrho A'As$ and $\varrho A'As \in R(A')$.

~~Where~~ $(\cdot)^{-1}$ denotes the inverse of an operator.

REMARK 2.

The corresponding theorem for interpolating splines ensures the existence of a unique element $\sigma \in S$ such that $A\sigma = z$.

The characterization of a smoothing spline which leads to its pratical evaluation in contained in the following corollary of Thorem 2.

COROLLARY (Characterization) ~~of~~ σ if smoothing spline corresponding to (T,A,z, ϱ) iff there exists $\lambda \in Z$ such that:

$$T'T\sigma = A'\lambda \quad ; \quad \lambda = \varrho (z - A\sigma) \tag{3.11}$$

Indeed, from eqs. (3.2) and (3.8):

$$L'L\sigma = L'p_0 \qquad ; \; L\sigma = [T\sigma, A\sigma]$$

$$\Rightarrow \quad T'T\sigma = \varrho A'(z - A\sigma)$$

from which eqs. (3.11) follow with $\lambda = \varrho(z - A\sigma)$. The first of eqs. (3.11) shows that $A'\lambda \in R(T') = N(T)^{\perp}$, where $(\;)^{\perp}$ denotes the orthogonal subspace, and the second equality follows from the properties of adjoint operators.

The estremal properties of the smoothing splines are condensed in the following theorem.

THEOREM 4. If z is an arbytrary but fixed element of Z and σ is the unique elemente of S satisfying the condition (3.10), then:

a) for any $s \in S$:

$$\| T(\sigma - s) \|_Y + \varrho \| A\sigma - z + \tfrac{1}{\varrho} Bs \|_Z = \qquad (3.12)$$

$$= \min_{x \in X} \left\{ \| T(x - s) \|_Y + \varrho \| Ax - z + \tfrac{1}{\varrho} Bs \|_Z \right.$$

and σ is the unique element of X having this property.

The minimum problem (3.1) follows from equation (3.12) when(s) is taken to be the null element of S.

REMARK 3.

The corresponding extremal properties of the interpolating splines [1, 2] are formally recovered by setting $\varrho = 0$ and B the null operator.

3.2 FINITE SPACE Z

For the case of specific interest here the number of "constraints" is finite. The space Z is consequently finite and, according to thorem I, existence is automatically guaranted whereas uniqueness requires that N(T) be also finite.

Suppose then that $Z \subset R^{\nu}$, with the usual inner product, and that N(T) is of dimensious q. Then [3]:

a) the operator A can be expressed as:

$$A x = [\langle K_1, x\rangle_X, ----, \langle K_n, x\rangle_X] \in R^n \qquad (3.13)$$

where the K_i are n independent linear continuous functionals on X;

b) if:

$$t = [t_1, t_2, ---, t_n] \in R^n$$

then:

$$A' t = \sum_{i=1}^{n} t_i K_i \qquad (3.14)$$

c) the following characterization theorem holds (see Corollary):

THEOREM 5. $T \in X$ is the smoothing spline corresponding to (T, K_i, z, ρ) iff there exist n coefficients $\bar{\lambda}_i$ such that:

$$T'T\sigma = \sum_{i=1}^{n} \bar{\lambda}_i K_i \in N(T)^{\perp}$$

$$\bar{\lambda}_i = \rho[z_i - \langle K_i, \sigma\rangle_X] \quad ; \quad i = 1, --, n \qquad (3.15)$$

REMARK 4.

The corresponding theorem for interpolating splines is formally recovered by replacing the last (n) conditions in eqs. (3.15) with the conditions $z_i = \langle K_i, \sigma\rangle_X$.

Upon the remark I the first equation characterizes the spline space S corresponding to (T, K_i). Hence:

$$B s = (A')^{-1} T'T s = \bar{\lambda} \in Z = R^n$$

and the extremal properties of the smoothing splines described by Theorem 4 can be formulated as:

THEOREM 6. If z in an arbitrary element of Z and σ is the unique element of S satisfying the condition $(3.15)_2$, then:

a) for any $x \in X$:

$$\|T(\sigma - x)\|_Y + \rho \sum_{i=1}^{n} [\langle K_i, x - \sigma\rangle_X^2] =$$

$$= \min_{s \in S} \left\{ \|T(s-x)\|_Y + \rho \sum_{i=1}^{n} \left[\langle K_i, x \rangle_X - z_i + \frac{\bar{\lambda}_i}{\rho} \right]^2 \right\} \qquad (3.16)$$

and any other $\bar{\sigma} \in S$ having this property belongs to the set:

$$\{\bar{\sigma}\} = \sigma + N(T)$$

b) for any $s \in S$:

$$\|T(\sigma - s)\|_Y + \rho \sum_{i=1}^{n} \left[\langle K_i, \sigma \rangle_X - z_i + \frac{\bar{\lambda}_i}{\rho} \right]^2 =$$

$$= \min_{x \in X} \left\{ \|T(x-s)\|_Y + \rho \sum_{i=1}^{n} \left[\langle K_i, x \rangle_X - z_i + \frac{\bar{\lambda}_i}{\rho} \right]^2 \right\} \qquad (3.17)$$

and σ is the unique element of X having this property.

The different classes of smoothing spline functions that can be obtained from the above abstract formulation depend on the choice of X, Y, T and K_i.

4. NORMAL CLOSED SMOOTHING SPLINES. EXISTENCE, UNIQUENESS AND CHARACTERIZA

Let C be a sufficiently smooth and regular closed contour. Take $X = H^q(C)$ and $Y = H^0(C)$ with their standard inner products. Denote by τ the curvilinear coordinate along C measured from an arbitrary point ω, and normalized with respect to the length of C.

For the operator $T : X = H^q(C) \to Y = H^0(C)$ take the q-th derivative D_q with respect to (τ). For $Z \subset R^n$ let the n functionals $K_i : H^q(C) \to R^n$ be defined by:

$$\langle K_i, x \rangle_{H^q} = x(\tau_i) = t_i \qquad 1 \le i \le n \qquad (4.1)$$

$$t = [t_1, t_2, \ldots, t_n] \in R^n$$

with

$$0 \le \tau_1 < \tau_2 < \cdots < \tau_n < 1$$

The null spaces of the operators A and T are then given by [1]:

$$N(A) = \{ x \in H^q(C) \mid \langle K_i, x \rangle_{H^q} = x(\tau_i) = 0 \ ; \ i \in [1, n] \}$$

$$N(T) = \{ x \in H^q(C) \mid x^{(q)} = 0 \} \equiv \{ x \in H^q(C) \mid x = \text{const.} \}$$

Hence their dimensions are equal to (n) and to (I), respectvely so that $N(T) \cap N(A) = \{0\}$ provided $n > 1$. The operator $L = H^q(C) \to P = H^q(C) \otimes H^0(C)$ is defined by:

$$L x = [D_q x, A x] = [D_q x, x(\tau_1), x(\tau_2), \ldots, x(\tau_n)] \qquad (4.2)$$

so that:

$$L\sigma - p_c = [D_q \sigma, x(\tau_1) - Z_1, \ldots, x(\tau_n) - Z_n] \qquad (4.3)$$

The minimum problem (3.1) reads:

$$\oint_C [\sigma^{(q)}]^2 d\tau + \rho \sum_{i=1}^{n} [\sigma(\tau_i) - t_i]^2 = \min_{x \in H^q(C)} \left\{ \oint_C [x^{(q)}]^2 d\tau + \rho \sum_{i=1}^{n} [x(\tau_i) - t_i]^2 \right\} \qquad (4.4)$$

and, according to theorem I, it has a unique solution, for any r, as long as $n > 1$.

The existence and uniqueness of $\sigma(\tau)$ is thus established.

Its characterization is accomplished through theorem 3.

From eq. (3.15) with $\bar{\lambda}_i = (-1)^q \lambda_i$; and from eq. (4.1) one has, subsequently:

$$< D_q' D_q \sigma, x >_{H^q} = \sum_{i=1}^{n} (-1)^q < \lambda_i K_i, x >_{H^q} = (-1)^q \sum_{i=1}^{n} \lambda_i x(\tau_i) \qquad \forall x \in H^q(C) \tag{4.5}$$

$$(-1)^q \lambda_i = \rho [z_i - \sigma(\tau_i)] \quad ; \quad 1 \le i \le n \tag{4.6}$$

where, according to theorem 5, the λ_i must satisfy the condition:

$$\sum_{i=1}^{n} < \lambda_i K_i, x >_{H^q} = 0 \qquad \forall x \in N(T) \tag{4.7}$$

stating that ($D_q' D_q \sigma$) must belong to $N(T)^{\perp}$.

As dim $N(T) = 1$, eq. (4.7) reduces only to the requirement that $\sum_{i=1}^{n} \lambda_i K_i$ be H^q orthogonal to unity. On account of eq. (4.5) this leads to:

$$\sum_{i=1}^{n} \lambda_i = 0 \tag{4.8}$$

The characterization of a function $\sigma^{(q)}$ satisfying eqs. (4.5) and (4.8) was developed in [1]

when dealing with closed interpolating splines.

This finding is consistent with the fact that, as mentioned in paragraph (3), the space of spline functions corresponding to (T, A) is the same, whether one deals with interpolating or smoothing splines [3].

The details will not be repeated here and only the relevant results will be stated.

The function $\sigma^{(q)}$ can be expressed as:

$$\sigma^{(q)}(\tau) = \sum_{j=0}^{q-1} \beta_{q+j} \frac{\tau^j}{j!} + \sum_{i=1}^{n} \lambda_i \frac{(\tau - \tau_i)_+^{q-1}}{(q-1)!} \tag{4.9}$$

where the $(n+q)$ coefficients β_{q+j} and λ_i satisfy the (q) equations [1]

$$\sum_{j=0}^{p-1} \left[\frac{\beta_{2q-p+j}}{(j+1)!} + (-1)^{q+j} \frac{\alpha_{p-j}}{j!} \right] = 0 \quad ; \quad 1 \le p \le q-1 \tag{4.10}$$

$$\sum_{i=1}^{n} \lambda_i = 0 \tag{4.11}$$

with:

$$\alpha_m = \sum_{i=1}^{n} \frac{\lambda_i \tau_i^m}{m!} \qquad ; \; m > 0$$

$$\alpha_m = 0 \qquad ; \; m \leq 0$$

Eqs 4.10 express the vanishing of the discontinuities of $\sigma^{(q)}$ and of its first ($q-2$) derivatives at the closure point Q, characterized by either values $\tau = 0^+$ or $\tau = 1^-$ of τ.

The function $\sigma^{(q)}$ given by eq. (4.9) is continuous on C together with its first ($q-2$) derivatives. The discontinuity:

$$\delta_i \, \sigma^{(2q-1)} = \sigma^{(2q-1)}(\tau_i^+) - \sigma^{(2q-1)}(\tau_i^-)$$

of the $q-1$)-th derivative of $\sigma^{(q)}$ at the point Q_i is given by:

$$\delta_i \, \sigma^{(2q-1)} = \lambda_i \tag{4.12}$$

The other n equations needed to compute the ($n+q$) coefficients β_{q+j} and λ_i are given by eqs. (4.6).

According to eqs. (3.6), (3.8) and (4.5) and the definition (3.5) of inner product in P the space of closed spline S_c is defined by:

$$S_c = \left\{ s \in H^q(C) \,/\, \langle D_q s, D_q x \rangle_{H^0} + \rho \sum_{i=1}^{n} [s(\tau_i) - t_i] \, x(\tau_i) = 0 ; \; \forall x \in H^q(C) \right\}$$

Hence, for any $x \in H^q(C)$

$$\oint_C s^{(q)} x^{(q)} d\tau + \rho \sum_{i=1}^{n} [s(\tau_i) - t_i] \, x(\tau_i) = 0$$

or, on account of eq. (4.6):

$$\oint s^{(q)} x^{(q)} d\tau = (-1)^q \sum_{i=1}^{n} \lambda_i x(\tau_i)$$

This last relation is the same as that derived in [1] and once again reflects the fact that the space S_c makes no reference to the type of closed spline, whether interpolating or smoothing.

The closed smoothing spline $\sigma \in S_c$ is obtained by integrating eq. (4.9) q times and using the arbitrary constants to impose the continuity of σ and of its first ($q-1$) derivatives at the closure point Q, so as to make $\sigma \in H^q(C)$.

The extremal properties formulated in section (1) follow from theoremen 6 with $T = D_q$ and $\bar{\lambda}_i = (-1)^q \lambda_i$ given by eq. (4.6).

This concludes the proof of the statements made in Section I .

5 . REFERENCES.

I. L. G. NAPOLITANO , V. LOSITO : " The closed spilnes functios " Comp; Meth; in Applied Mechanics and Engineering, Vol.I3 (I978) pp. 335-350 .

2. L. G. NAPOLITANO : " Hermite Closed Spile Functions " Aerotecnica , Missili e Spazio , Vol.7 No I (I978) pp. 30-36.

3. J.P. LAURENT : "Approximation et Optimitation ",Herman,Paris,I972.